Unit III

Sequences and Their Limits

A sequence in a set S is a function whose domain is the set $\mathbb{N}$ of natural numbers, and whose range is contained in the set S. In this chapter, we will be concerned with sequences in $\mathbb{R}$ and will discuss what we mean by the convergence of these sequences.

Definition A **sequence of real numbers** (or a **sequence in** $\mathbb{R}$) is a function defined on the set $\mathbb{N} = \{1, 2, \ldots\}$ of natural numbers whose range is contained in the set $\mathbb{R}$ of real numbers.

In other words, a sequence in $\mathbb{R}$ assigns to each natural number $n = 1, 2, \ldots$ a uniquely determined real number. If $X : \mathbb{N} \to \mathbb{R}$ is a sequence, we will usually denote the value of X at n by the symbol x_n rather than using the function notation $X(n)$. The values x_n are also called the **terms** or the **elements** of the sequence. We will denote this sequence by the notations

$$X, \quad (x_n), \quad (x_n : n \in \mathbb{N}).$$

Of course, we will often use other letters, such as $Y = (y_k)$, $Z = (z_i)$, and so on, to denote sequences.

seems evident. For example, we may define the sequence of reciprocals of the even numbers by writing

$$X := \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \cdots\right),$$

though a more satisfactory method is to specify the formula for the general term and write

$$X := \left(\frac{1}{2n} : n \in \mathbb{N}\right)$$

or more simply $X = (1/2n)$.

Introduction to Infinite Series

In elementary texts, an infinite series is sometimes "defined" to be "an expression of the form"

(1) $$x_1 + x_2 + \cdots + x_n + \cdots.$$

However, this "definition" lacks clarity, since there is *a priori* no particular value that we can attach to this array of symbols, which calls for an *infinite* number of additions to be performed.

Definition If $X := (x_n)$ is a sequence in $\mathbb{R}$, then the **infinite series** (or simply the **series**) **generated by** X is the sequence $S := (s_k)$ defined by

$$\begin{aligned}
s_1 &:= x_1 \\
s_2 &:= s_1 + x_2 \quad (= x_1 + x_2) \\
&\cdots \\
s_k &:= s_{k-1} + x_k \quad (= x_1 + x_2 + \cdots + x_k) \\
&\cdots
\end{aligned}$$

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The numbers x_n are called the **terms** of the series and the numbers s_k are called the **partial sums** of this series. If lim S exists, we say that this series is **convergent** and call this limit the **sum** or the **value** of this series. If this limit does not exist, we say that the series S is **divergent.**

It is convenient to use symbols such as

$$\sum(x_n) \quad \text{or} \quad \sum x_n \quad \text{or} \quad \sum_{n=1}^{\infty} x_n$$

to denote both the infinite series S generated by the sequence $X = (x_n)$ and also to denote the value lim S, in case this limit exists.

Examples Consider the sequence $X := (r^n)_{n=0}^{\infty}$ where $r \in \mathbb{R}$, which generates the **geometric series**:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots + r^n + \cdots.$$

We will show that if $|r| < 1$, then this series converges to $1/(1-r)$. Indeed, if $s_n := 1 + r + r^2 + \cdots + r^n$ for $n \geq 0$, and if we multiply s_n by r and subtract the result from s_n, we obtain (after some simplification):

$$s_n(1-r) = 1 - r^{n+1}.$$

Therefore, we have $$s_n - \frac{1}{1-r} = \frac{r^{n+1}}{1-r},$$

from which it follows that $$\left| s_n - \frac{1}{1-r} \right| \leq \frac{|r|^{n+1}}{|1-r|}$$

Since $|r|^{n+1} \to 0$ when $|r| < 1$, it follows that the geometric series converges to $1/(1-r)$ when $|r| < 1$.

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INFINITE SERIES:

Definition: Let $\{u_n\}$ be a sequence of real numbers.

i.e. $\{u_1, u_2, u_3, u_4, \ldots\ldots\ldots, u_n, \ldots\ldots\ldots\infty\}$ be a sequence of real numbers.
Then, the expression $u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots\infty$ [i.e. the sum of the terms of the sequence, which are infinite in number] is called an **infinite series** and is denoted by $\sum_{n=1}^{\infty} u_n$ or more briefly, by $\sum u_n$.

n^{th} Term: The term u_n in an infinite series is called the n^{th} term of the series.

Finite series:

If the number of terms in the series is limited, then the series is called a ***finite series.***

Infinite series:

If the number of terms in the series is infinite, then the series is called an ***infinite series.***

n^{th} Partial sum:

The sum of the first n terms of the series is called its ***nth partial sum*** of $\sum u_n$ and is denoted by S_n.

i.e., $S_n = u_1 + u_2 + \ldots\ldots + u_n$.

Clearly, S_n is a function of n and as n increases indefinitely, then the following three cases arise:

(i) If S_n tends to a finite limit as $n \to \infty \Rightarrow \sum u_n$ is ***convergent.***

(ii) If S_n tends to $+\infty$ or $-\infty$ as $n \to \infty \Rightarrow \sum u_n$ is ***divergent.***

(iii) If S_n does not tends to a unique limit as $n \to \infty \Rightarrow \sum u_n$ is ***oscillatory or non-convergent.***

Remark: Any series which diverge or oscillate are said to be non-convergent series.

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Q.No.1.: Examine the behavior of the series: $1+\frac{1}{2}+\frac{1}{2^2}+......+\frac{1}{2^{n-1}}+\frac{1}{2^n}+......$.

Sol.: Given series is $1+\frac{1}{2}+\frac{1}{2^2}+......+\frac{1}{2^{n-1}}+\frac{1}{2^n}+......$

Here $S_n = 1+\frac{1}{2}+\frac{1}{2^2}+......+n^{th}$ term. $\left[\because \text{it is a G.P. whose first term is 1 and common ratio is } 1/2. \therefore S_n = \frac{a(1-r^n)}{1-r}\right]$

$$\Rightarrow S_n = \frac{1\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} \Rightarrow S_n = 2\left(1-\frac{1}{2^n}\right).$$

Taking limit on both sides, we get

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left(1-\frac{1}{2^n}\right) = 2(1-0) = 2.$$

$\Rightarrow$ The given series converges to 2.

Q.No.2.: Examine the behavior of the series: $1+2+3+.......+n+........\infty$.

Sol.: Given series is $1+2+3+.......+n+........\infty$.

Here $S_n = 1+2+3+.........+n = \frac{n(n+1)}{2}$.

Taking limit on both sides, we get

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} \to \infty.$$

$\Rightarrow$ The given series is divergence.

Comparison Test:

Statement: Let $\sum u_n$ and $\sum v_n$ be two positive term series, such that from and after some particular term, $\lim_{n\to\infty} \frac{u_n}{v_n} = k$ (a non-zero, finite quantity), then $\sum u_n$ and $\sum v_n$ either both converge or both diverge, or in other words, both the series behave alike.

Remark: To select auxiliary series (or Harmonic series) $\sum v_n = \sum \frac{1}{n^p}$, it should be noted that p = difference in degree of n in denominator and numerator of u_n.

3. p-series Test:

(Behaviour of Auxiliary Series or Harmonic Series)

Statement: Show that the series $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+......+\frac{1}{n^p}+...... = \sum \frac{1}{n^p}$

(i) converges if $p > 1$ and

(ii) diverges if $p \le 1$.

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Q.No.1. Discuss the behaviour of the series $\sum_{n=1}^{\infty} \frac{n+3}{n^3-n+1}$.

Sol.: Here $u_n = \frac{n+3}{n^3-n+1}$. Take $v_n = \frac{1}{n^2}$.

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1+\frac{3}{n}}{1-\frac{1}{n^2}+\frac{1}{n^3}} = 1$ (a finite, non-zero number).

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^2} \approx \sum \frac{1}{n^p}$ (here $p = 2 > 1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent.

Q.No.2.: Discuss the behaviour of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$.

Sol.: Here $u_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$. Take $v_n = \frac{1}{\sqrt{n}}$.

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}$ (a finite, non-zero number).

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^{1/2}} \approx \sum \frac{1}{n^p}$ (here $p = 1/2 < 1$) is divergent. [by p-series test]

Hence $\sum u_n$ is also divergent.

D'Alembert's Ratio Test:

Statement: If $\sum u_n$ be a positive term series s.t. from and after some particular term,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = k, \text{ then } \sum u_n$$

(i) converges if $k < 1$ and

(ii) diverges if $k > 1$.

This test fails when limit does not exist or equal to 1.

Here $\frac{u_{n+1}}{u_n}$ measures the rate or growth of the terms of the series.

Note: This test was developed by Jean Le-Rond D'Alembert (1717–1783) a French Mathematician.

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Remark: Ratio test fails when k = 1. So when k = 1, in that case comparison test is helpful in determining the behaviour of the series.

Counter example:

For the convergent series, $\sum_{n=1}^{\infty} \frac{1}{n^2}$: $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n^2}{(1+n)^2} = 1$

Also for the divergent series, $\sum_{n=1}^{\infty} \frac{1}{n}$: $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n}{(1+n)} = 1$

Thus ratio test cannot be used to distinguish between convergent and divergent series when k = 1

Q.No.1.: Discuss the behaviour of infinite series $\sum \frac{n!}{n^n}$.

Sol.: Here $u_n = \frac{n!}{n^n}$ and so $u_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}}$.

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{(n+1)}} \times \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1. \quad \text{[since e=2.71828...]}$$

Hence $\sum u_n$ is convergent (by D'Alembert's Ratio Test).

Q.No.2.: Discuss the behaviour of infinite series $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \ldots\ldots$.

Sol.: Here $u_n = \frac{n^p}{n!}$ and so $u_{n+1} = \frac{(n+1)^p}{(n+1)!}$.

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^p}{(n+1)!} \times \frac{n!}{n^p} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^p . \frac{1}{(n+1)} = (1)^p.0 = 0 < 1.$$

Hence $\sum u_n$ is convergent (by D'Alembert's Ratio Test).

Q.No.3.: Discuss the behaviour of infinite series $\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \ldots\infty \quad (x > 0)$.

Sol.: Here $u_n = \frac{x^n}{(2n-1)(2n)}$ and so $u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$.

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\left(2-\frac{1}{n}\right)(2)}{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)} .x = x.$$

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Case 1: The given series is convergent if $x < 1$.

Case 2: The given series is divergent if $x > 1$.

Case 3: When $x = 1$, $u_n = \frac{1}{(2n-1)(2n)}$. Take $v_n = \frac{1}{n^2}$.

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(2-\frac{1}{n}\right)2} = \frac{1}{4} \text{ (a finite non-zero number)}.$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^2} \approx \sum \frac{1}{n^p}$ (here $p = 2 > 1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent when $x = 1$.

Hence the given series converges if $x \le 1$ and diverges if $x > 1$.

RAABE'S TEST*:

Statement: If $\sum u_n$ be a positive term series s.t. from and after some particular term, $\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = k$ or $\lim_{n\to\infty} n\left[1 - \left|\frac{u_{n+1}}{u_n}\right|\right] = k$,

then $\sum u_n$ **(i)** converges if $k > 1$ and

(ii) diverges if $k < 1$.

Remarks: (i) Raabe's test fails when $k = 1$.

(ii) This test is very useful when D'Alembert's ratio test fails.

Q.No.1.: Discuss the behaviour of infinite series $\sum \frac{4.7.10.......(3n+1)}{1.2.3.......n} x^n$.

Sol.: Here $\frac{u_n}{u_{n+1}} = \frac{4.7.......(3n+1)}{1.2.......n} x^n \times \frac{1.2....n.(n+1)}{4.7....(3n+1).(3n+4)} \frac{1}{x^{n+1}} = \frac{(n+1)}{(3n+4)} \cdot \frac{1}{x}$

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{3x}$. Hence, by D'Alembert's ratio test

$\sum u_n$ (i) converges if $\frac{1}{3x} > 1$ and (ii) diverges if $\frac{1}{3x} < 1$.

i.e. $\sum u_n$ (i) converges if $x < \frac{1}{3}$ and (ii) diverges if $x > \frac{1}{3}$.

But this test fails if $x = \frac{1}{3}$. Now if $\lim_{n\to\infty} \frac{u_n}{u_{n+1}}$ does not involve e, then we can apply Raabe's

Now let us try the Raabe's test. When $x = \frac{1}{3}$, then

$$\frac{u_n}{u_{n+1}} = \left[1+\frac{1}{n}\right]\left[1+\frac{4}{3n}\right]^{-1} = \left[1+\frac{1}{n}\right]\left[1-\frac{4}{3n}+\frac{16}{9n^2}\ldots\ldots\right] = 1-\frac{1}{3n}+\frac{4}{9n^2}-\ldots\ldots$$

$$\Rightarrow \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right] = \lim_{n\to\infty}\left[-\frac{1}{3}+\frac{4}{9n}-\ldots\ldots\right] = -\frac{1}{3} < 1.$$

$\Rightarrow$ By Raabe's test, the given series diverges when $x = \frac{1}{3}$.

Hence the given series $\sum u_n$ (i) converges if $x < \frac{1}{3}$ and (ii) diverges if $x \geq \frac{1}{3}$.

Q.No.2.: Discuss the behaviour of infinite series

$$1+\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\frac{3.6.9.12}{7.10.13.16}x^4+\ldots\ldots\ldots\infty.$$

Sol.: Here (by omitting first term) $u_n = \frac{3.6.9.....(3n)}{7.10.13.......(3n+4)}x^n$

and $\therefore u_{n+1} = \frac{3.6.9........(3n)(3n+3)}{7.10.13.......(3n+4)(3n+7)}x^{n+1}$.

Now $\frac{u_n}{u_{n+1}} = \frac{3.6.9.......(3n)}{7.10.13........(3n+4)}x^n \times \frac{7.10.13......(3n+4)(3n+7)}{3.6.9.....(3n).(3n+3)}\frac{1}{x^{n+1}} = \frac{(3n+7)}{(3n+3)}.\frac{1}{x}$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(1+\frac{7}{3n}\right)}{\left(1+\frac{1}{n}\right)}.\frac{1}{x} = \frac{1}{x}.$$

Hence by D'Alembert's ratio test:

$\sum u_n$ (i) converges if $\frac{1}{x} > 1$ and (ii) diverges if $\frac{1}{x} < 1$.

i.e. $\sum u_n$ (i) converges if $x < 1$ and (ii) diverges if $x > 1$.

But this test fails if $x = 1$. Now if $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}$ does not involve e, then we can apply Raabe's

test or Gauss's test. Now let us try the Raabe's test.

When $x = 1$, then

$$\frac{u_n}{u_{n+1}} = \left[1+\frac{7}{3n}\right]\left[1+\frac{1}{n}\right]^{-1} = \left[1+\frac{7}{3n}\right]\left[1-\frac{1}{n}+\frac{1}{n^2}\ldots\ldots\right] = 1+\frac{4}{3n}-\frac{4}{3n^2}-\ldots\ldots$$

$$\Rightarrow \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right] = \lim_{n\to\infty}\left[\frac{4}{3}-\frac{4}{3n}-\ldots\ldots\right] = \frac{4}{3} > 1.$$

$\Rightarrow$ By Raabe's test, the given series converges when $x = 1$.

Hence the given series $\sum u_n$ (i) converges if $x \leq 1$ and (ii) diverges if $x > 1$.

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Q.No.3.: Discuss the behaviour of infinite series $\frac{x}{3}+\frac{1.2}{3.5}x^2+\frac{1.2.3}{3.5.7}x^3+......\infty\ (x>0)$.

Sol.: Here $u_n=\frac{1.2.3....n}{3.5.7....(2n+1)}x^n$ and so $u_{n+1}=\frac{1.2.3....n(n+1)}{3.5.7....(2n+1)(2n+3)}x^{n+1}$

Now $\frac{u_n}{u_{n+1}}=\frac{1.2.3....n}{3.5.7...(2n+1)}x^n\times\frac{3.5.7....(2n+1)(2n+3)}{1.2.3....n(n+1)}\frac{1}{x^{n+1}}=\frac{(2n+3)}{(n+1)}\frac{1}{x}=\frac{2\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{n}\right)}\frac{1}{x}$.

$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{2\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{n}\right)}\frac{1}{x}=\frac{2}{x}$.

Hence by D'Alembert's ratio test

$\sum u_n$ (i) converges if $\frac{2}{x}>1$ and (ii) diverges if $\frac{2}{x}<1$.

i.e. $\sum u_n$ (i) converges if $x<2$ and (ii) diverges if $x>2$.

But this test fails if $x=2$. Now if $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}$ does not involve e, then we can apply Raabe's test or Gauss's test. Now let us try the Raabe's test.

When $x=2$ then

$$\frac{u_n}{u_{n+1}}=\left[1+\frac{3}{2n}\right]\left[1+\frac{1}{n}\right]^{-1}=\left[1+\frac{3}{2n}\right]\left[1-\frac{1}{n}+\frac{1}{n^2}......\right]=1+\frac{1}{2n}-\frac{1}{2n^2}-......$$

$$\Rightarrow \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right]=\lim_{n\to\infty}\left[\frac{1}{2}-\frac{1}{2n}-.....\right]=\frac{1}{2}<1.$$

$\Rightarrow$By Raabe's test, the given series diverges when $x=1$.

Hence the given series $\sum u_n$ (i) converges if $x<2$ and (ii) diverges if $x\geq 2$.

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Fourier Series

Introduction

Most of the events in nature and many other systems, being periodic in nature. In many engineering problems, especially in the study of periodic phenomena in conduction of heat, electro-dynamics and acoustic, it is necessary to express a function in a series of sines and cosines. Most of the single-valued functions which occur in applied mathematics can be expressed in the form

$$\frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \dots\dots\dots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \dots\dots$$

where $a_0, a_1, a_2, a_3, \dots\dots\dots, b_1, b_2, b_3, \dots\dots\dots\dots$ are real constants and the series is known as Fourier series. The constants $a_0, a_1, a_2, a_3, \dots\dots, b_1, b_2, b_3, \dots\dots$ are called Fourier's Coefficients of the periodic function.

Euler's Formulae

The Fourier series for the function f(x) in the interval $\alpha<x<\alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{(i)}$$

Determination of the Fourier's Constants:

To find a_0: assume that (i) can be integrated from x = c to x = c+2π, term by term, so that we have

$$\int_C^{C+2\pi} f(x)\,dx = \frac{a_0}{2}\int_C^{C+2\pi} 1\,dx + \int_C^{C+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)dx + \int_C^{C+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)dx$$

$$= \frac{a_0}{2}(x)_C^{C+2\pi} + 0 + 0$$

$$= \frac{a_0}{2}(C + 2\pi - C) = \pi a_0$$

or $$a_0 = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\,dx$$

To find a_n: multiplying each side of (i) by cosnx and integrate with respect to x between the limits x = C to x = C + 2π, so that we have

$$\int_C^{C+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_C^{C+2\pi}\cos nx\,dx + \int_C^{C+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\cos nx\,dx + \int_C^{C+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\cos nx\,dx$$

$$= 0 + \int_C^{C+2\pi}\sum_{n=1}^{\infty} a_n \cos^2 nx\,dx + 0$$

$$= a_n \pi$$

or $$a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx\,dx$$

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To find b_n multiply both side of (i) by sinnx and integrate with respect to x between the limits x = c to x = C+2π, so that we have

$$\int_C^{C+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}\int_C^{C+2\pi}\sin nx\,dx + \int_C^{C+2\pi}\left(\sum_{n=1}^{\infty} a_n\cos nx\right)\sin nx\,dx + \int_C^{C+2\pi}\left(\sum_{n=1}^{\infty} b_n\sin nx\right)\sin nx\,dx$$

$$= 0 + 0 + \int_C^{C+2\pi}\sum_{n=1}^{\infty} b_n\sin^2 nx\,dx = b_n\pi$$

$$b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin nx\,dx$$

The values $a_o = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx$$

$$b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin nx\,dx$$

are called Euler's formulae

Euler's Formulae for Different Intervals

Case (i): If C = 0, then the interval for the above series (i) become $0<x<2\pi$ and the Euler's formulae reduce to

$$a_o = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Example 1. Find a Fourier series to represent $x-x^2$ from $x = -\pi$ to $x = \pi$ and hence deduce

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\dots\dots$$

Solution. Let $x - x^2 = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$ (1)

Then $a_o = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x)^2\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} = -\frac{2\pi^2}{3}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx$$

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$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} - (1-2x) - \left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{-4(-1)^n}{n^2} \qquad \because \cos n\pi = (-1)^n$$

$$\therefore a_1 = 4/1^2, a_2 = -4/2^2, a_3 = 4/3^2, a_4 = -4/4^2 \text{ etc}$$

Finally

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(-\frac{\cos nx}{n}\right) - (1-2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= -2(-1)^n/n$$

$$\therefore b_1 = 2/1, b_2 = -2/2, b_3 = 2/3, b_4 = -2/4 \text{ etc}$$

Substituting the values of a_o, a_n, b_n in (i) we get

$$x - x^2 = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\ldots\right]$$

$$+ 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\ldots\right]$$

Example Obtain the Fourier Series expansion for the function

$f(x) = x^2, -\pi<x<\pi$

Hence, deduce that

(a) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{6}$

(b) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12}$

(c) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$

Solution: We have $f(x) = x^2$

$f(x)$ is an even function, therefore, $f(x)$ contains only cosine terms. Hence $b_n = 0$

Let $f(x) = x^2 = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ (1)

Then we have $a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{3\pi}\left[\pi^3 - (-\pi)^3\right]$$

$$= \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$$

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and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{2}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + \frac{2\pi}{n^2}\cos n\pi + 0\right]$$

$$= \frac{2}{\pi}\,\frac{2\pi}{n^2}(-1)^n = \frac{4}{n^2}(-1)^n$$

Substituting in (1), we get

i.e. $x^2 = \frac{\pi^2}{3} + 4\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx\right]$

$$= \frac{\pi^2}{3} + 4\left[\frac{(-1)^1}{1^2}\cos x + \frac{(-1)^2}{2^2}\cos 2x + \frac{(-1)^3}{3^2}\cos 3x + \ldots\right]$$

$$= \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\right]$$

or $x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\right]$

At $x = \pi$ and $x = 0$, the function f(x) is continuous

Putting $x = \pi$, is (2), we get

$$\pi^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos \pi}{1^2} - \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} - \ldots\right]$$

or $\pi^2 = \frac{\pi^2}{3} - 4\left[-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \ldots\right]$

$$\pi^2 - \frac{\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$$

or $4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] = \frac{2\pi^2}{3}$

i.e. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$

putting $x = 0$, in (2), we get

$$0 = \frac{\pi^2}{3} - 4\left[\frac{\cos 0}{1^2} - \frac{\cos 0}{2^2} + \frac{\cos 0}{3^2} - \ldots\right]$$

or $0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right]$

or $4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right] = \frac{\pi^2}{3}$

or $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$ (4)

Adding (3) and (4) we get $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

Unit III

FUNCTIONS HAVING POINT OF DISCONTINUITY

In deriving Euler's formulae for the constants a_o, a_n and b_n we have assumed that f(x) is continuous in the given interval. In some cases f(x) may have a finite number of discontinuities. We can also express such functions as Foruier series. For example, consider a function f(x) defined as follows:

$$f(x) = f_1(x), \qquad c<x<x_o$$
$$= f_2(x), \qquad x_o<x<c+2\pi$$

Where x_o is a point of discontinuities for f(x) is the interval (c, c+2π) and $\lim_{x\to o-} f(x)$ i.e. $f(x_o - o)$ and $\lim_{x\to o+} f(x)$ i.e. $f(x_o + o)$ exist unequal and are finite. We determine the values of a_o, a_n and b_n can be computed as

$$a_o = \frac{1}{\pi}\left[\int_c^{x_o} f_1(x)\,dx + \int_{x_o}^{C+2\pi} f_2(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_o} f_1(x)\cos nx\,dx + \int_{x_o}^{c+2\pi} f_2(x)\cos nx\,dx\right]$$

$$\text{and } b_n = \frac{1}{\pi}\left[\int_c^{x_o} f_1(x)\sin nx\,dx + \int_{x_o}^{c+2\pi} f_2(x)\sin nx\,dx\right]$$

At the point of discontinuity i.e. at $x = x_o$ the Fourier series converges to

$$\frac{1}{2}[f(x_o + o) + f(x_o - o)] = \frac{1}{2}(FB + FC)$$

If f(x) satisfies Dirichlet's conditions and f(x) is expressed as

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos nx + \sum_1 b_n \sin nx$$

in the interval {C, C+2π], then f(x) converges to $f(x_o)$ if x_o is a point of continuity of f(x) in the given interval

Thus from above we conclude that

(i) It may be seen from the graph, that at a point of finite discontinuity $x = x_o$ there is a finite jump equal to BC in the value of the function f(x) at $x = x_o$

(ii) A given function f(x) may be defined by different formulae in different regions. Such types of functions are quite common in Fourier series

(iii) At a point of discontinuity the sum of the series is equal to mean of the limits on the right and left.

Example Find Fourier series for the function defined by

$$f(x) = \begin{cases} -1 & \text{for } -\pi<x<o \\ 0 & \text{for } x = o \\ 1 & \text{for } 0<x<\pi \end{cases}$$

hence proved that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \ldots..$$

Solution: By Fourier series, we have

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{Then } a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{o}(-1)\,dx + \int_{o}^{\pi} 1\,dx\right]$$

$$= \frac{1}{\pi}\left[(-x)_{-\pi}^{o} + (x)_{o}^{\pi}\right]$$

$$a_o = \frac{1}{\pi}[-\pi + \pi] = 0$$

$$\text{and } a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{o} -\cos nx\,dx + \frac{1}{\pi}\int_{o}^{\pi}\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}[\sin nx]_{-\pi}^{o} + \frac{1}{n}[\sin nx]_{o}^{\pi}\right\}$$

$$= \frac{1}{\pi}[-o + o] = 0$$

$$\text{Finally, } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{o} -\sin nx\,dx + \frac{1}{\pi}\int_{o}^{\pi}\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\frac{1}{n}[\cos nx]_{-\pi}^{o} - \frac{1}{n}[\cos nx]_{o}^{\pi}\right\}$$

$$= \frac{1}{n\pi}[1 - \cos n\pi - \cos n\pi + 1] = \frac{1}{n\pi}[2 - 2\cos n\pi] = \frac{2}{n\pi}[1-(-1)^n]$$

$$\Rightarrow b_n = \frac{2}{n\pi}[1-(-1)^n],\ n = 1, 2, 3, \ldots\ldots$$

If n is even $b_n = 0$ if n is odd $b_n = \frac{4}{n\pi}$ Hence from (1) we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + \ldots\ldots\ldots$$

$$f(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \ldots\ldots \quad (2)$$

The expansion (2) is required Fourier expansion putting $x = \frac{\pi}{2}$ is (2) we get

$$1 = \frac{4}{\pi}\left[\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \ldots\ldots\right] = \frac{4}{\pi}\left[1 + \frac{1}{3}(-1) + \frac{1}{5}(1) + \frac{1}{7}(-1) + \ldots\right]$$

$$= \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\right] = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Unit III

HALF RANGE SERIES

Many a time it is required to obtain a Fourier expansion of a function f(x) for the range (0, C) which is half the period of the Fourier series. As it is immaterial whatever the function may be outside the range 0<x<c, we extend the function to cover the range –c<x<c so that the new function may be odd or even. The extension of the functions period being made in such a way that their graphs became either symmetrical to the axis of y or symmetrical to origin, and then the expansion contains either only the cosine terms along with a_o or only the sine terms. Thus, we may get the different forms of series for the same functions.

Sine Series: If we have to expand a function f(x) as a sine series in 0 to C, then we expand the function first from –C to C and then the make the reflection at the origin, so that f(x) = –f(–x) then the expanded function became odd and will give the required Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{C}$$

$$\text{Where } b_n = \frac{2}{C} \int_0^C f(x) \sin \frac{n\pi x}{C}\, dx$$

Cosine Series: If it is required to express f(x) as a cosine series in 0<x<C, then first we expand the function from –C to C so that its reflection became about the axis of y i.e., the graph became symmetrical about the axis of y then the expanded Fourier Cosine series contains

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{C}$$

$$\text{where } a_o = \frac{2}{C} \int_0^C f(x)\, dx$$

$$\text{and } a_n = \frac{2}{C} \int_0^C f(x) \cos \frac{n\pi x}{C}\, dx$$

Example . Expand for f(x) = k for 0<x<2 in a half range (i) Sine series (ii) cosine series

Solution. f(x) = k and C = 2

(i) $$b_n = \frac{2}{C} \int_0^C f(x) \sin \frac{n\pi x}{C} \text{ in half range } (0, C)$$

$$= \frac{2}{2} \int_0^C k \sin \frac{n\pi x}{2}\, dx$$

$$= k\, \frac{2}{n\pi} \left(-\cos \frac{n\pi x}{2}\right)_o^2 = \frac{2k}{n\pi} [-\cos n\pi + 1]$$

Half range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$$

$$k = \sum_{n=1}^{\infty} \frac{2k}{n\pi}[1-\cos n\pi] \sin\frac{n\pi x}{2} = \frac{2k}{\pi}\sum_{n=1}^{\infty}[1-(-1)^n] \sin\frac{n\pi x}{2}$$

$$\Rightarrow k = \frac{2k}{\pi}\left[2\sin\frac{\pi x}{2} + 2\sin\frac{3\pi x}{2} + 2\sin\frac{5\pi x}{2} + \ldots\right]$$

(ii) $$a_o = \frac{2}{C}\int_0^C f(x)\,dx = \frac{2}{2}\int_0^2 k\,dx = k\,(x)_0^2 = 2k$$

$$a_n = \frac{2}{C}\int_0^C f(x)\cos\frac{n\pi x}{C}\,dx = \frac{2}{2}\int_0^2 k\cos\frac{n\pi x}{2}\,dx$$

$$= k\,\frac{2}{n\pi}\left[\sin\frac{n\pi x}{2}\right]_0^2 = \frac{2k}{n\pi}\sin n\pi$$

$$\Rightarrow a_n = 0$$

Therefore, from

$$f(x) = \frac{a_o}{2} + \sum a_n \cos\frac{n\pi x}{2} \text{ we have}$$

$$f(x) = k = \frac{1}{2}(2k) + \sum_{n=1}^{\infty} 0\cos\frac{n\pi x}{2}$$

$$\Rightarrow f(x) = k$$